

Exponentiation of multiparticle amplitudes in scalar theories.

II. Universality of the exponent

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Abstract

It has been shown recently that the amplitude of the creation of n real scalar particles by one virtual boson near n -particle threshold exhibits exponential behavior at $n \sim 1/\lambda$. We extend this result to the processes of multiparticle production at threshold by two virtual bosons. We find that both the tree-level amplitude and leading- n loop corrections have the same exponential behavior, with the common exponent for $1 \rightarrow n$ and $2 \rightarrow n$ processes, for various kinematics. This result strongly indicates that the exponent for multiparticle amplitudes is independent of the initial state and that there may exist a semiclassical approach to the study of multiparticle production.

I. INTRODUCTION

A reliable calculation of multiparticle amplitudes remains an interesting problem in quantum field theory. The problem exists in most weakly coupled theories [1–9], including the simplest case of $\frac{\lambda}{4}\varphi^4$ model, where it has been observed that the tree amplitude to produce n final particles from an initial virtual one exhibits a factorial dependence on the number of outgoing particles like $n!\lambda^{n/2}$, which leads to an unacceptably rapid growth of the transition rate at the tree level at large enough n , $n \sim 1/\lambda$. Since the contribution from the first loop is of order λn^2 as compared to the tree-level result, and from the k -th loop one expects a contribution of order $(\lambda n^2)^k$, it is clear that the calculation of the amplitude at $n \sim 1/\lambda$ requires the summation of the whole perturbation series. Recently, it has been demonstrated that at the n -particle threshold, leading in n contributions from each loop sum up to exponent [10]

$$A_{1 \rightarrow n} = A_{1 \rightarrow n}^{\text{tree}} \cdot e^{B\lambda n^2} \quad (1)$$

where B is some numerical constant and

$$A_{1 \rightarrow n}^{\text{tree}} = n! \left(\frac{\lambda}{8} \right)^{\frac{n-1}{2}} \quad (2)$$

is the tree amplitude (the scalar boson mass is set equal to 1). Equation (1) determines correctly the threshold amplitude with the account of all loops at $n \sim 1/\sqrt{\lambda}$, in which case the loops have the same order of magnitude as the tree-level contribution. More importantly, at $n \sim 1/\lambda$, Eq. (1) provides strong indication for the exponentiation of the loop corrections. In other words, one expects that at $n \sim 1/\lambda$, the threshold amplitude has the following form,

$$A(1 \rightarrow n) \cong \sqrt{n!} \cdot e^{\frac{1}{\lambda} F(\lambda n)} \quad (3)$$

where F is some function that has the following expansion at small λn ,

$$F(\lambda n) = F_{\text{tree}} + B\lambda^2 n^2 + O(\lambda^3 n^3)$$

with

$$F_{\text{tree}} = \frac{\lambda n}{2} \ln \frac{\lambda n}{8} - \frac{\lambda n}{2} \quad (4)$$

so at $\lambda n \ll 1$, Eq. (3) reduces to Eq. (1). The behavior of the function F is presently unknown at $\lambda n \sim 1$.

It is likely that the exponential form of the amplitude, Eq. (3), is a consequence of the semiclassical nature of the processes of multiparticle production. So, one may hope that there exists some semiclassical-type technique for calculating the amplitudes of these processes.

In this direction, one interesting suggestion [3,11,12] is to try to generalize the Landau method for the calculation of semiclassical matrix elements from one-dimensional quantum mechanics to field theory. The Landau method [13] is a powerful technique which allows one to calculate the matrix elements of almost any regular operator between two semiclassical states, with different energies, of a particle moving in one-dimensional potential. In field theory, the direct analogues of these matrix elements are the multiparticle amplitudes: for instance, the amplitude to produce n scalar bosons at threshold by a virtual particle can be written in the form $\langle n|\phi|0\rangle$, where $|n\rangle$ is the state with n particles at threshold.

One specific feature of the semiclassical matrix elements in one-dimensional quantum mechanics which is captured by the Landau method is that they do not depend, to the exponential precision, on the operator for which they are calculated; they depend only on the states between which the operator is sandwiched, as well as on the details of the potential. Therefore, if there really exists a generalization of the Landau method to field theory, one would expect that the exponents of the matrix elements $\langle n|\hat{O}|0\rangle$ should not depend on the operator \hat{O} , provided the latter does not depend on the coupling constant λ . In other words, the multiparticle amplitudes should be the same, to the exponential precision, for all few-particle initial states. So, a check of the existence of the Landau-type procedure for calculating multiparticle amplitudes will be the demonstration that they are indeed independent of the initial state. Note that similar conjecture has been made for instanton-induced processes [14,15].

Presently, it is unclear how to perform reliable calculations in the regime $\lambda n \sim 1$. However, the technique of Ref. [10] can be generalized to deal with initial states that may contain more than one virtual particle. In this way one should be able to verify that the exponential part of the tree level amplitudes is the same as in Eq. (4) for all few-particle (containing much less than $1/\lambda$ particles) initial states, and that the leading- n loop corrections exponentiate to $\exp(B\lambda n^2)$ with exactly the same coefficient B as in Eq. (1).

In this paper we perform this check for initial states that contain two virtual bosons and final states containing n real bosons at rest ($\mathbf{p}_f = 0$). Since there are two incoming particles, the energy can be distributed arbitrarily between them. The first case that we consider is the amplitude $2 \rightarrow n$ integrated over 4-momentum of one initial particle. This quantity is equal to the matrix element $\langle n|\varphi^2|0\rangle$ at n -particle threshold. In two other cases we evaluate the amplitude of scattering of two particles with 4-momenta (E, \mathbf{p}) and $(n - E, -\mathbf{p})$ into n particles at rest (we set the mass of the boson equal to 1) in two different regimes: at $E \ll n$, so that the energy of one initial particle is much smaller than that of the other, and at E of order n when the energy of the two initial particles are comparable to each other (we will take the spatial momentum \mathbf{p} of the incoming particle to be small, $|\mathbf{p}| \sim 1$). In the latter case we consider the regime when E does not belong to the interval $(0, n)$, so one initial particle carries negative energy (it may appear more natural to view this particle as an outgoing one; this is of course a matter of terminology: this particle is virtual anyway, and we are free to call it incoming). The interval $0 < E < n$ is peculiar: for these values of E even the tree-level amplitude has singular behavior as a function of E , which is a consequence of the threshold kinematics, and does not have a well defined limit in the large n , fixed E/n regime. Due to this peculiarity, the case $0 < E < n$ is not suitable for our main purpose.

The technique that we will apply is a direct extension of one developed in Ref. [10]. In all three cases, the calculations show that the tree level amplitude and the leading- n loop corrections exponentiate to the same exponent as in the case of $1 \rightarrow n$ process. This result strongly indicates that the exponent of multiparticle amplitudes in fact do not depend on

the initial state and supports the existence of some semiclassical, possibly Landau-type, approach to the study of multiparticle production.

The paper is organized as follows. In Section II we will briefly review the approach developed in Ref. [10] for summing leading- n loop corrections. In Section III, this technique is applied for calculating matrix elements of the operators φ^2 . Section IV is devoted to $2 \rightarrow n$ amplitudes in the case of one soft initial particle, while in Section V the case of two hard initial particles is considered. Section VI contains concluding remarks. Appendix is devoted to a brief discussion of the case $0 < E < n$, and the tree and one-loop results are presented.

II. GENERAL FORMALISM

We consider φ^4 theory with unbroken discrete symmetry in $(d+1)$ dimensional spacetime with the Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \quad (5)$$

Hereafter we set the mass of the boson m to be equal to 1.

For calculating multiparticle amplitudes at threshold, there exists a convenient formalism [4] which reduces the problem to the calculation of Feynman graphs in certain classical background. In Ref. [4] this formalism has been developed for the $1 \rightarrow n$ amplitude, but it is easy to extend it for treating processes with two incoming particles.

Let us outline briefly this technique (see Ref. [4] for details). Consider a transition from an initial virtual particle with $(d+1)$ -momentum $P_\mu = (n, \mathbf{0})$ into n final particles, each with $(d+1)$ -momentum $(1, \mathbf{0})$. The reduction formula for the amplitude can be written in the following form,

$$\langle n | \phi(x) | 0 \rangle = \lim_{\varepsilon_0^2 \rightarrow 1} \lim_{J_0 \rightarrow 0} (\varepsilon_0^2 - 1)^n \frac{\partial^n}{\partial J_0^n} \langle 0 | \varphi(x) | 0 \rangle_{J=J_0 \exp(i\varepsilon_0 t)} \quad (6)$$

where the matrix element is calculated in the presence of a source $J = J_0 \exp(i\varepsilon_0 t)$.

Taking the limits $\varepsilon_0 \rightarrow 0$, $J_0 \rightarrow 0$ simultaneously, one can show that A_n is determined by

$$A_n = \frac{\partial^n}{\partial z_0^n} \langle 0 | \varphi | 0 \rangle |_{z=0} \quad (7)$$

where the expectation value $\langle 0 | \varphi | 0 \rangle$ is calculated in the following classical background

$$\varphi_0(t) = \frac{z_0 e^{it}}{1 - \frac{\lambda}{8} z_0 e^{2it}} \quad (8)$$

which is a solution to the field equation.

It is straightforward to generalize this technique to the case of matrix elements of arbitrary operators. In particular, the matrix element of the operator φ^2 can be obtained by differentiating its vacuum expectation value in the presence of the background field φ_0 ,

$$\langle n | \varphi^2 | 0 \rangle = \frac{\partial^n}{\partial z_0^n} \langle 0 | \varphi^2 | 0 \rangle |_{z_0=0} \quad (9)$$

Analogously, to calculate the amplitude of scattering of two initial particles with momenta (E, \mathbf{p}) and $(n-E, -\mathbf{p})$ into n bosons at threshold, one should differentiate the corresponding full propagator n times with respect to z_0 ,

$$A_{2 \rightarrow n}(E, \mathbf{p}) = \frac{\partial^n}{\partial z_0^n} \int d^{d+1}x d^{d+1}y e^{iEx^0 + i(n-E)y^0} e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} \mathcal{D}(x, y) \quad (10)$$

where $\mathcal{D}(x, y)$ is the two-point Green function calculated in the classical background φ_0 . It is convenient to use the mixed coordinate-momentum representation,

$$\mathcal{D}_{\mathbf{p}}(x^0, y^0) \equiv \int d^d\mathbf{x} d^d\mathbf{y} e^{i\mathbf{p}(\mathbf{y}-\mathbf{x})} \mathcal{D}(x, y)$$

so Eq. (10) can be rewritten in the following form,

$$A_{2 \rightarrow n}(E, \mathbf{p}) = \frac{\partial^n}{\partial z_0^n} \int dx^0 dy^0 e^{iEx^0 + i(n-E)y^0} \mathcal{D}_{\mathbf{p}}(x^0, y^0). \quad (11)$$

So, the amplitudes to produce n final particles at threshold can be obtained by differentiating the corresponding Green functions (“generating functions”), calculated in the presence of the background field φ_0 , n times with respect to the parameter of the background, z_0 .

The Green functions which enter the right hand sides of Eqs. (9) and (11), in their turn, can be computed in the perturbation theory around the classical background φ_0 . It is convenient to introduce the Euclidean time variable,

$$\tau = it + \frac{1}{2} \ln \frac{\lambda}{8} + \ln z_0 + \frac{i\pi}{2}, \quad (12)$$

in terms of which the background field has the form

$$\varphi_0(\tau) = -i\sqrt{\frac{2}{\lambda}} \frac{1}{\cosh \tau}.$$

Note that the background field has a singularity at $\tau = i\pi/2$. By expanding the Lagrangian around the background φ_0 one obtains the Feynman rules shown in Fig. 1.

Obviously, perturbative calculations become more and more complicated at higher loops. However, if one is interested only in the leading- n contribution from each loop level, considerable simplification occurs [10], which allows for the summation of the whole perturbative series. The key point is that, at each loop level, the large- n behavior of the amplitude depends only on the structure of the singularity at $\tau = i\pi/2$ of the generating functions (in our case $\langle 0|\varphi^2|0\rangle$ or $\mathcal{D}_{\mathbf{p}}(x^0, y^0)$). Thus, our strategy is to find the generating function near the singularity and after that recover the multiparticle amplitude.

In the case of $1 \rightarrow n$ process, the procedure has been developed [10] for obtaining the leading singularity of the corresponding generating function at $\tau = i\pi/2$ at any given loop order, which reduces the problem to the calculation of tree graphs in some effective theory. We will apply this technique to the case of the processes with two initial particles. Let us begin with the matrix element of φ^2 .

III. MATRIX ELEMENTS OF φ^2

Let us first discuss the Feynman graphs that contribute to the generating function, $\langle 0|\varphi^2|0\rangle$. There is one tree-level graph shown in Fig. 2. At the one-loop level, there are two different types of graphs. The two-loop graphs are much more numerous, only one of them is presented in Fig. 3a.

As explained above, at each loop level it is sufficient to calculate only the leading singularity of the generating function. The latter can be found by extending the technique of Ref. [10], that reduces the problem to the evaluation of tree graphs. Since this extension is straightforward, let us summarize here only the prescription (see Ref. [10] for details):

- For a given loop graph one should cut some propagator lines in such a way that the resulting graph is tree and connected. In cases when there are various ways to cut, the result is given by the sum over all possibilities. The number of lines to be cut is equal to the number of loops.
- For each propagator line, say, $G(\tau, \tau')$, that has been cut, one attributes the factors $B^{1/2}\lambda\varphi^2(\tau)$ and $B^{1/2}\lambda\varphi^2(\tau')$ to the two vertices that this line connects, i.e., at τ and τ' , respectively. The constant B is defined by

$$B = \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{9}{8\omega_{\mathbf{p}}(\omega_{\mathbf{p}}^2 - 1)(\omega_{\mathbf{p}}^2 - 4)}$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + 1}$. All propagators that have not been cut should be replaced by the operator $(\partial_\tau^2 + 3\lambda\varphi_0^2)^{-1}$. Therefore, the tree graphs obtained by our cutting procedure do not contain spatial momenta explicitly.

Let us demonstrate how this prescription works for a particular graph shown in Fig. 3a. This graph has a symmetry factor of $1/4$. Since it is a two-loop graph, the number of propagators to be cut is 2. There are in fact 5 possibilities to cut in such a way that the graph remains connected: one can cut the lines (1,3), (2,3), (1,4), (2,4), or (3,4) (one cannot cut two propagators (1,2), since then the graph will become disconnected). In fact, one can see that the first four ways lead to essentially the same graphs, while the graph obtained in the fifth case has another topology. As a result, one obtains a sum of two graphs with symmetry factors of 1 and $1/4$, as shown in Fig. 3b. The black circles (“bullets”) represent the factors $B^{1/2}\lambda\varphi^2$.

By analyzing the tree graphs obtained by cutting the initial loop graphs, Figs. 2 and 3, one can see that they have the form of a tree cascade starting from two initial lines corre-

sponding to the two operators $\varphi(\tau)$ (a two-branch tree). Each branch contains contributions of φ_0 , $B^{1/2}\lambda\varphi_0^2$ and higher power of λ . One observes (analogously to the $1 \rightarrow n$ case) that the same series of graphs is obtained when one solves the field equation without the mass term,

$$\partial_\tau^2 \varphi + \lambda \varphi^3 = 0 \quad (13)$$

perturbatively with respect to λ , with the condition that the first two terms in the expansion over λ are $\varphi_0 + B^{1/2}\lambda\varphi_0^2$.

Eq. (13) has the following exact solution

$$\varphi_{\text{cl}} = -\sqrt{\frac{2}{\lambda}} \frac{1}{\tau - \tau_0} \quad (14)$$

which has a single pole at $\tau = \tau_0$. One can fix τ_0 by requiring that the solution has the expansion $\varphi_{\text{cl}} = \varphi_0 + B^{1/2}\lambda\varphi_0^2 + \dots$. As a result, one obtains

$$\tau_0 = i\pi/2 - \sqrt{2B\lambda}.$$

Then φ_{cl} can be represented in the following form,

$$\varphi_{\text{cl}} = \varphi_0 \sum_k (B^{1/2}\lambda\varphi_0)^k \quad (15)$$

At first sight, the generating function is equal to φ_{cl}^2 , which has the following expansion,

$$\varphi_{\text{cl}}^2 = \varphi_0^2 \sum_{k=0}^{\infty} (k+1) (B^{1/2}\lambda\varphi_0)^k \quad (16)$$

However, this is not true. First, one notes that all graphs obtained by the cutting procedure contain even number of bullets, so there should be only even powers of $B^{1/2}\lambda$ in the series for the generating function. Second, from every tree graph with $2l$ bullets one can reconstruct $(2l)!/(2^l l!)$ graphs of the original theory at l -loop order by pairing the bullets into propagator lines. So, to recover the generating function one should omit in Eq. (16) all terms with odd k and for terms with even k , $k = 2l$, one should multiply the coefficient by the factor $(2l)!/(2^l l!)$. In this way one obtains,

$$\langle 0|\varphi^2|0\rangle = \varphi_0^2 \sum_{l=0}^{\infty} \frac{(2l+1)!}{2^l l!} (\lambda^2 B \varphi_0^2)^k \quad (17)$$

To obtain the matrix element $\langle n|\varphi^2|0\rangle$ one substitutes Eq. (8) into Eq. (17) and differentiates n times with respect to z . Recalling Eq. (12), one obtains the following result,

$$\langle 0|\varphi^2|n\rangle = \frac{n!n}{2} \left(\frac{\lambda}{8}\right)^{\frac{n}{2}-1} \sum_{k=0}^{\infty} \frac{(\lambda B n^2)^k}{k!} = A_{\varphi^2}^{\text{tree}} e^{B\lambda n^2}$$

where

$$A_{\varphi^2}^{\text{tree}} = \frac{n!n}{2} \left(\frac{\lambda}{8}\right)^{\frac{n}{2}-1}$$

can be identified with the matrix element calculated at the tree level, which differs from the tree $1 \rightarrow n$ amplitude (2) only by a pre-exponential factor proportional to n . The leading- n loop contributions exponentiate to the same factor of $\exp(B\lambda n^2)$. Therefore, the exponent for the matrix element of φ^2 coincides with the $1 \rightarrow n$ amplitude, which is the desired result.

IV. AMPLITUDES $2 \rightarrow n$ WITH ONE SOFT INITIAL PARTICLE

In this section we consider the process of scattering of two initial virtual particles, among which one is soft (with energy $E \ll n$) and the other is hard, into n final particles at threshold (the spatial momenta of initial particles are assumed to be small, i.e. of order 1).

It is convenient to rewrite Eq. (11) in a slightly modified form. One notes that it is sufficient to consider only one integral over dy^0 , the integral over dx^0 results in the delta-function of energy conservation, which we will not explicitly write in what follows. Furthermore, one introduces the notation $z(t) = z_0 e^{it}$, and goes to Euclidean time, then the $2 \rightarrow n$ amplitude is written as follows,

$$A_{2 \rightarrow n}(E, \mathbf{p}) = \frac{\partial^n}{\partial z^n} G(\tau)$$

where

$$G(\tau) = e^{E\tau} \int d\tau' \mathcal{D}_{\mathbf{p}}(\tau, \tau') e^{-E\tau'} \quad (18)$$

We will call $G(\tau)$ the generating function for the $2 \rightarrow n$ amplitude. At the tree level this result was obtained firstly in [16] by direct summation of graphs. Let us begin with discussing the tree level.

A. Tree level

At the tree level, it is easier to find the amplitude and then recover the generating function. The tree-level amplitude can be found directly from the exact tree propagator in the background field φ_0 , Eq. (8) [17,18]

$$D_{\mathbf{p}}(\tau, \tau') = \frac{1}{W_{\mathbf{p}}} (f_1^\omega(\tau) f_2^\omega(\tau') \theta(\tau' - \tau) + f_2^\omega(\tau) f_1^\omega(\tau') \theta(\tau - \tau'))$$

where

$$W_{\mathbf{p}} = 2\omega(\omega^2 - 1)(\omega^2 - 4), \quad \omega = \sqrt{\mathbf{p}^2 + 1}$$

and

$$f_1^\omega(\tau) = e^{\omega\tau} \left(\frac{12}{(1 + e^{2\tau})^2} + \frac{6(\omega - 2)}{1 + e^{2\tau}} + (\omega - 1)(\omega - 2) \right)$$

$$f_2^\omega(\tau) = f_1^{-\omega}(\tau)$$

To evaluate the integral (18), one expands the functions f_1 and f_2 in series,

$$f_1^\omega(\tau) = e^{\omega\tau} \sum_{k=0}^{\infty} (-1)^k f_{1k} e^{2k\tau}$$

$$f_2^\omega(\tau) = e^{-\omega\tau} \sum_{k=0}^{\infty} (-1)^k f_{2k} e^{2k\tau} \tag{19}$$

where

$$f_{1k} = \delta_{k0}(\omega - 1)(\omega - 2) + 6\omega + 12k$$

$$f_{2k} = \delta_{k0}(\omega + 1)(\omega + 2) - 6\omega + 12k$$

The tree amplitude $2 \rightarrow n$ can be now obtained by direct calculation. One finds,

$$A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) = n! \left(\frac{\lambda}{8} \right)^{n/2} \frac{1}{W_{\mathbf{p}}} \sum_{k=0}^{n/2} \left(\frac{f_{1k} f_{2(\frac{n}{2}-k)}}{2k + \omega - E} - \frac{f_{2k} f_{1(\frac{n}{2}-k)}}{2k - \omega - E} \right) \quad (20)$$

It is worth noting that $A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p})$ develops a series of poles at $E = 2k \pm \omega$. The physical reason for these poles is that for these values of E the tree graphs may include a propagator with on-shell momentum (see Fig. 4).

To obtain the asymptotic behavior of the amplitude at large n , one notes that at $E \sim 1$, the sum in Eq. (20) is saturated by a finite number of terms with $k \sim 1$. So we can replace $f_{1,2(n/2-k)}$ by $6n$ and extend the sum to $k = \infty$. We obtain,

$$A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) = 6n!n \left(\frac{\lambda}{8} \right)^{n/2} \frac{1}{W_{\mathbf{p}}} \sum_{k=0}^{\infty} \left(\frac{f_{1k}}{2k + \omega - E} - \frac{f_{2k}}{2k - \omega - E} \right)$$

Note that the sum in this equation converges and does not depend on n . Let us introduce the function,

$$C(E, \mathbf{p}) = \frac{3}{2} \frac{1}{W_{\mathbf{p}}} \sum_{k=0}^{\infty} \left(\frac{f_{1k}}{2k + \omega - E} - \frac{f_{2k}}{2k - \omega - E} \right)$$

The tree amplitude can now be rewritten in the form

$$A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) = 4n \cdot n! \left(\frac{\lambda}{8} \right)^{n/2} C(E, \mathbf{p}) \quad (21)$$

Note that the tree amplitude is proportional to the tree matrix element of φ^2 ,

$$A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) = \lambda C(E, \mathbf{p}) \langle n | \varphi^2 | 0 \rangle_{\text{tree}}$$

and correspondingly differs from the tree $1 \rightarrow n$ amplitude (2) only by a factor of n . Therefore the generating function is proportional to that of the operator φ^2 ,

$$G^{\text{tree}}(\tau) = \lambda C(E, \mathbf{p}) \varphi_0^2(\tau) \quad (22)$$

One can verify by differentiation that the generating function (22) indeed gives rise to the amplitudes (21) at large n .

B. Loop corrections

To sum leading- n loop corrections to the $2 \rightarrow n$ amplitude, one should calculate the generating function

$$G(\tau) = e^{E\tau} \int d\tau' \mathcal{D}(\tau, \tau') e^{-E\tau'}$$

in the region close to singularity. The technique that we apply here is the same as that used in the case of the operator φ^2 : one cuts every loop graph so that it becomes tree and connected and attaches bullets to the cut lines. In this way one obtains the leading singularity of the propagator $\mathcal{D}(\tau, \tau')$ as a sum of graphs that typically have the form shown in Fig. 5. One can see that these graphs are the same as those obtained by expanding the propagator in the background field φ_{cl}

$$D_{\text{cl}}(\tau, \tau') = (-\partial_\mu^2 + 1 + 3\lambda\phi_{\text{cl}})^{-1}$$

in the perturbation series in λ , where the background field φ_{cl} and its expansion are given by Eqs. (14) and (15), respectively. The only difference is that only graphs with even number of bullets is present for \mathcal{D} and the symmetry coefficients differ by a factor of $(2l)!/(2^l l!)$, where $2l$ is the number of bullets.

The quantity

$$G_{\text{cl}}(\tau) = e^{E\tau} \int d\tau' D_{\text{cl}}(\tau, \tau') e^{-E\tau'} \quad (23)$$

can be easily calculated since the fields φ_{cl} and φ_0 have the same behavior around their singularities, while the singularities are located at different points. Recalling Eq. (22) one writes

$$\begin{aligned} G_{\text{cl}}(\tau) &= \lambda C(E, \mathbf{p}) \varphi_{\text{cl}}^2(\tau) \\ &= \lambda C(E, \mathbf{p}) \varphi_0^2 \sum_{l=0}^{\infty} (k+1) (\lambda^2 B \varphi_0^2)^k \end{aligned} \quad (24)$$

$G(\tau)$ can be obtained from this expression by dropping all terms in the sum with odd k and correcting the coefficients by the factors $(2l)!/(2^l l!)$. Comparing with Eq. (17), one finds that the generating function in our case is proportional to that of the matrix element of φ^2 ,

$$G(\tau) = \lambda C(E, \mathbf{p}) \langle 0 | \varphi^2 | 0 \rangle$$

so the loop corrections exponentiate in the same way as for $\langle n | \varphi^2 | 0 \rangle$,

$$A_{2 \rightarrow n}(E, \mathbf{p}) = A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) e^{B\lambda n^2}$$

So, we have established that both the tree expression and the leading- n loop corrections for the amplitude $2 \rightarrow n$ in the case when one initial particle is soft sum up to $\exp(\frac{1}{\lambda} F_{\text{tree}} + B\lambda n^2)$.

V. AMPLITUDES $2 \rightarrow n$ WITH TWO HARD INITIAL PARTICLES

In this section we consider the case when both initial particles have energies of order n . For the amplitude to have a regular limit at large n , we take the energy of one incoming particle, E , to be negative, while the energy of the other is larger than n . The case when energies of both initial particles are positive is briefly discussed in Appendix.

A. Tree level

The tree amplitude in our case can be determined from its representation in terms of the sum in Eq. (20). However, from Eq. (20) it is not straightforward to extract its asymptotic behavior in the regime $n \rightarrow \infty$, $E/n = \text{fixed}$. We adopt here another approach. We will find the generating function from the equation that it obeys,

$$(-\partial_\tau^2 + 3\lambda\varphi_0^2) \left(e^{-E\tau} G^{\text{tree}}(\tau) \right) = e^{-E\tau} \quad (25)$$

(we have omitted the term ω^2 which is inessential near the singularity). Since the amplitude to produce n particles depends on the details of the behavior of generating function in the

region $|\tau - i\pi/2| \sim 1/n$, and $E \sim n$, we look for the solution of Eq. (25) in the region $|\tau - i\pi/2| \sim 1/E$. Near the singularity, Eq. (25) reduces to

$$\left(-\partial_\tau^2 + \frac{6}{(\tau - i\pi/2)^2}\right) \left(e^{-E\tau} G^{\text{tree}}(\tau)\right) = e^{-E\tau} \quad (26)$$

Eq. (26) can be solved exactly, the solution has the following form

$$G^{\text{tree}}(\tau) = -\frac{1}{5E^2} \left(\frac{6}{E^2(\tau - i\pi/2)^2} + \frac{6}{E(\tau - i\pi/2)} + 3 + E(\tau - i\pi/2) - E^2(\tau - i\pi/2)^2 - \right. \\ \left. - E^3(\tau - i\pi/2)^3 e^{E\tau} \int_{-\infty}^{\tau} \frac{e^{-E\tau'}}{\tau' - i\pi/2} d\tau' \right) \quad (27)$$

We have kept in Eq. (27) not only the leading singular term, but all terms which are of order E^{-2} at $E|\tau - \pi/2| \sim 1$. Those terms are equally important for the evaluation of the amplitude: while the leading singular term (the first term in parenthesis in Eq. (27)) gives rise to the contribution of order nE^{-4} , the second term produces E^{-3} , etc. (see below). Note that $E < 0$, so the integral in this equation converges.

The large- n asymptotics of the amplitude is uniquely determined by the behavior of the generating function which has been found in Eq. (27). To recover the amplitude one should find a series in $e^{2\tau}$ which has the same behavior around the singularity as the r.h.s. of Eq. (27). Apparently there are various ways to write down the series in $e^{2\tau}$ that coincides with Eq. (27) near the singularity, but they all yield the same result for the amplitude. Technically, the most convenient way to write the series is to make the following replacement,

$$\tau - i\pi/2 \rightarrow -\frac{1 + e^{2\tau}}{2} \quad (28)$$

which is valid near the singularity, so the generating function obtains the form

$$G^{\text{tree}}(\tau) = -\frac{1}{5E^2} \left(\frac{24}{E^2(1 + e^{2\tau})^2} - \frac{12}{E(1 + e^{2\tau})} + 3 - \frac{E}{2}(1 + e^{2\tau}) - \frac{E^2}{4}(1 + e^{2\tau})^2 \right. \\ \left. - \frac{E^3}{4}(1 + e^{2\tau})^3 e^{E\tau} \int_{-\infty}^{\tau} \frac{e^{-E\tau'}}{(1 + e^{2\tau'})} d\tau' \right)$$

which is indeed a series in $e^{2\tau}$. Recalling Eq. (12), the tree amplitude can be derived by differentiating with respect to z . One obtains,

$$A_{2 \rightarrow n}^{\text{tree}} = -\frac{12n!}{5} \left(\frac{\lambda}{8}\right)^{\frac{n}{2}} \frac{1}{E^4(n-E)^4} \left((n-E)^5 + E^5\right) \quad (29)$$

Again, the exponential part of the amplitude is equal to $\exp(\lambda^{-1} F_{\text{tree}}(\lambda n))$, while the energy dependence enters only the pre-exponential factor.

B. Loop corrections

In analogy to the case of one soft initial particle, one should first calculate G_{cl} , Eq. (23). To do this we recall that D_{cl} is the propagator in the background field φ_{cl} which has the pole at $\tau = i\pi/2 - \sqrt{2\lambda B}$. So, instead of Eq. (26) one has

$$\left(-\partial_\tau^2 + \frac{6}{(\tau - i\pi/2 + \sqrt{2\lambda B})^2}\right) (e^{-E\tau} G_{\text{cl}}(\tau)) = e^{-E\tau}$$

Since this equation has the same form as Eq. (26), its solution is given by a formula analogous to Eq. (27), with the only difference that the pole at $i\pi/2$ is shifted to $i\pi/2 - \sqrt{2\lambda B}$. Making the replacement (28), one obtains the solution in the following form (we do not write explicitly the terms that are regular at the singularity)

$$G(\tau) = -\frac{1}{5E^2} \left(\frac{24}{E^2 (1 + e^{2\tau} - 2\sqrt{2\lambda B})^2} + \frac{12}{E (1 + e^{2\tau} - 2\sqrt{2\lambda B})} \right. \\ \left. - \frac{E^3}{4} (1 + e^{2\tau} - 2\sqrt{2\lambda B})^3 e^{E\tau'} \int_{-\infty}^{\tau} \frac{e^{-E\tau'}}{(1 + e^{2\tau} - 2\sqrt{2\lambda B})} d\tau' \right) + \text{regular terms} \quad (30)$$

which should be differentiated with respect to z in order to obtain the amplitude. Remembering that we should omit terms with odd powers of $B^{1/2}$ and multiply the coefficient of $(B^{1/2})^{2l}$ by $(2l)/(2^l l!)$, we obtain after a tedious but straightforward calculation,

$$A_{2 \rightarrow n} = A_{2 \rightarrow n}^{\text{tree}} \sum_{k=0}^{\infty} \frac{(\lambda B n^2)^k}{k!} = A_{2 \rightarrow n}^{\text{tree}} e^{\lambda B n^2}$$

where $A_{2 \rightarrow n}^{\text{tree}}$ is given by Eq. (29). Therefore, although in the case of two hard initial particles the calculations are more complicated, the result is the same as in the case of the matrix element of φ^2 or when one initial particle is soft: leading- n corrections sum up to the exponent $\exp(B\lambda n^2)$.

VI. CONCLUSION

We have studied the amplitude of the processes $2 \rightarrow n$ in the φ^4 theory by the technique that allows to sum up all leading- n loop corrections at n -particle threshold. We have shown that the $2 \rightarrow n$ amplitudes, regardless of how the initial energy is distributed among the two initial particles (except for some peculiar cases), coincide with the amplitude $1 \rightarrow n$ to the exponential precision, at least when only leading- n loop contributions are taken into account. The similar result can be easily obtained in the case of broken discrete symmetry. Our results, though not being a rigorous proof, strongly support the hypothesis that the amplitude of multiparticle processes is independent, in the exponential approximation, of the initial few-particle state. The picture here is similar to that of quantum mechanics, where the calculation of semiclassical matrix elements by the Landau method requires no knowledge on the details of the operator sandwiched between the semiclassical states. Our calculations, therefore, indicate that there may exist an extension of the Landau method to field theory, which, hopefully, could bring about the understanding of multiparticle amplitudes.

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APPENDIX A: $2 \rightarrow n$ AMPLITUDES FOR $0 < E < n$

In this Appendix we will briefly consider the behavior of the tree amplitude $2 \rightarrow n$ in the case when both initial particles have positive energies of order n , and also present the result for the one-loop correction to this amplitude.

The tree amplitude can be derived by making use of Eq. (20). At $E \sim n$ and $0 < E < n$, the sum in Eq. (20) is saturated by the terms with $k \approx E/2$. For these values of k one has $f_{1,2k} \approx 6E$, $f_{1,2(n/2-k)} \approx 6(n-E)$. The sum in Eq. (20) can be extended so that k runs from $-\infty$ to ∞ , and one writes,

$$\begin{aligned} A_{2 \rightarrow n}^{\text{tree}}(E, \mathbf{p}) &= n! \left(\frac{\lambda}{8} \right)^{n/2} \frac{36E(n-E)}{W_{\mathbf{p}}} \sum_{k=-\infty}^{\infty} \left(\frac{1}{2k + \omega - E} - \frac{1}{2k - \omega - E} \right) \\ &= n! \left(\frac{\lambda}{8} \right)^{n/2} \frac{72E(n-E)}{\pi W_{\mathbf{p}}} \left(\cot \frac{\pi}{2}(E + \omega) - \cot \frac{\pi}{2}(E - \omega) \right) \end{aligned}$$

Note that the amplitude has a very rapid oscillating behavior as a function of E , so it has no regular limit in the regime $n \rightarrow \infty$, $E/n = \text{fixed}$, but rather the amplitude depends on, say, the fractional part of $E/2$. Due to this behavior of the amplitude at the tree level, one should not, in general, expect that the loop corrections will sum up into a regular factor like $e^{B\lambda n^2}$.

We have performed the calculation of the one-loop correction to the amplitude by direct evaluation of the three graphs shown in Fig. 6. At large n , the result is

$$\begin{aligned} A^{1\text{-loop}} &= n! \left(\frac{\lambda}{8} \right)^{n/2} \frac{18}{\pi} \lambda E(n-E) \left[\frac{4B}{W_{\mathbf{p}}} \left(E^2 + (n-E)^2 \right) \left(\cot \frac{\pi}{2}(E + \omega) - \cot \frac{\pi}{2}(E - \omega) \right) + \right. \\ &\quad \left. + 9E(n-E) \int \frac{d\mathbf{k}}{W_{\mathbf{k}} W_{\mathbf{p}-\mathbf{k}}} \left(\cot \frac{\pi}{2}(E + \omega_{\mathbf{k}} + \omega_{\mathbf{p}-\mathbf{k}}) - \cot \frac{\pi}{2}(E - \omega_{\mathbf{k}} + \omega_{\mathbf{p}-\mathbf{k}}) \right) \right] \quad (\text{A1}) \end{aligned}$$

One sees that the one-loop correction is rather complicated and in general is not equal to $B\lambda n^2 \cdot A^{\text{tree}}$, unlike the cases considered in the body of this paper. The only exception is the theory in (2+1) dimensions, $d = 2$. In that case the integral in Eq. (A1) is infrared divergent at $\mathbf{k} = 0$ and $\mathbf{k} = \mathbf{p}$, the same is true for B . If one introduces an infrared cutoff p_0

(say, by considering final particles not exactly at threshold but with finite momenta of order p_0), Eq. (A1) reduces to $A^{\text{1-loop}} = B\lambda n^2 \cdot A^{\text{tree}}$. This result can be easily understood since in (2+1) dimensions, the loop corrections near threshold are dominated by the rescattering of final particles. The analysis then is completely similar to the case of $1 \rightarrow n$ processes, and further details can be found in Ref. [10].

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FIGURES

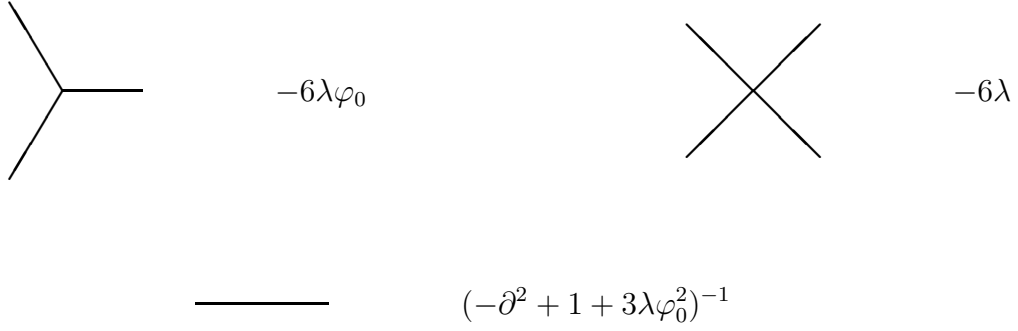


FIG. 1. Feynman rules for calculating multiparticle amplitudes

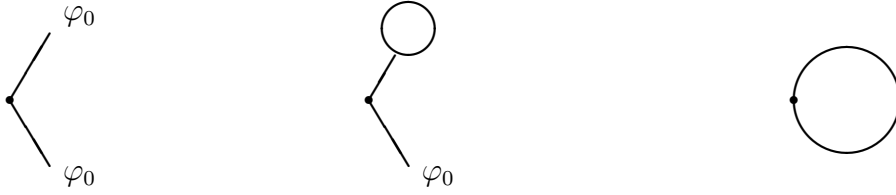


FIG. 2. The tree and one-loop graphs which contribute to the matrix element $\langle 0|\varphi^2|0\rangle$.

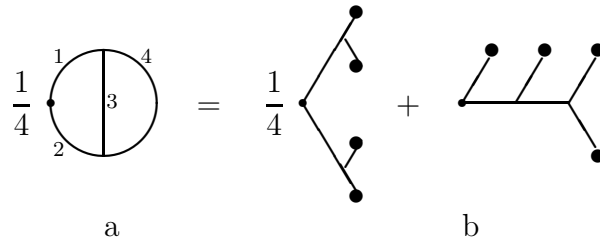


FIG. 3. The two-loop graph for the operator φ^2 (a) and a representation (b) of its leading singularity

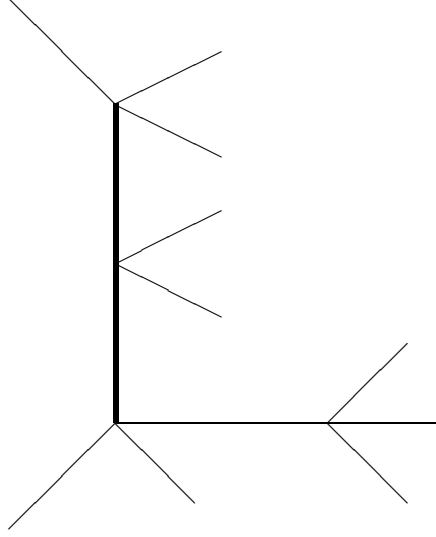


FIG. 4. A typical tree graph for the process $2 \rightarrow n$ (ordinary Feynmann rules are assumed). When the momentum running along one of the thick lines is on-shell, the tree amplitude is infinite.

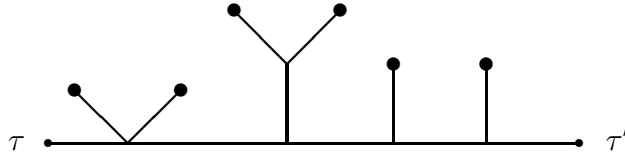


FIG. 5. A typical graph in the perturbative expansion of $\mathcal{D}(\tau, \tau')$

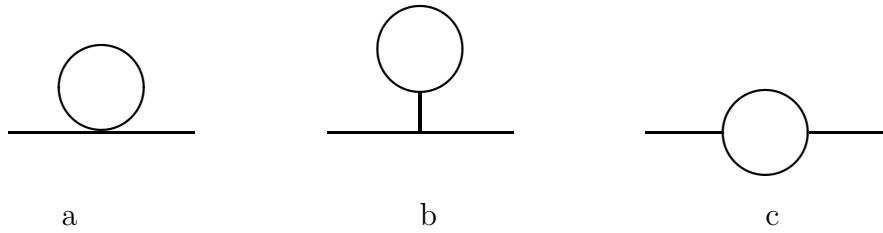


FIG. 6. The one-loop graphs contributing to $\mathcal{D}(x, y)$